KPZ Equation and Surface Growth Model

Masato Hisakado

Department of Pure and Applied Sciences,
University of Tokyo,
3-8-1 Komaba, Meguro-ku, Tokyo, 153, Japan

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Abstract

We consider the ultra-discrete Burgers equation. All variables of the equation are discrete. We classify the equation into five regions in the parameter space. We discuss behavior of solutions. Using this equation we construct the deterministic surface growth models respectively. Furthermore we introduce noise into the ultra-discrete Burgers equation. We present the automata models of the KPZ equation. One model corresponds to the discrete version of the ASEP and the other to the Kim-Kosterlitz model.

1 Introduction

The noisy Burgers equation appears in a variety of problems in non-equilibrium statistical mechanics.[1] Denoting the velocity field by $u(x,t)$ the equation reads

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x} \quad + \text{noise}. \tag{1.1}$$

Burgers studied the noiseless equation with random initial data. We are interested in the case the noise is of the form $\partial\xi/\partial x$, to conserve u locally. Then (1.1) is a prototype for a driven diffusive system. If we set $h = \int u dx$, (1.1) becomes

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + (\frac{\partial h}{\partial x})^2 + \xi. \tag{1.2}$$

We can interpret $h(x,t)$ as the height of a (one dimensional) surface. (1.2) is called the Karder-Parisi-Zhang (KPZ) equation.[2] This equation governs the shape fluctuations of the various growth model.[3]

In this paper we investigate the Burgers equation and the surface growth model without noise and with noise. With noise, the Burgers equation describes steady growth in the long time limit, however, without noise, it becomes the relaxation of an initially rough surface to the flat surface.

The relations between the Burgers equation and the surface growth model as studied from the view point of the universality classes, because the surface growth models are the discrete and the Burgers equation is continuous. We study the direct relations between discrete and continuous using the ultra-discrete method. Recently Nagatani, Nishinari and Takahahi presented the ultra-discrete version of the Burgers equation .[4],[5] We study solutions of the this equation. Behavior of solutions can be classified into five regions in the parameter space. The solutions can be understood from the viewpoint of creation and annihilation of particles and anti-particles. Using behavior of particles and anti-particles we can construct the surface models. The surface model corresponds to the deterministic version of the KPZ equation. Further more we present ultra-discrete version of the noisy Burgers equation and the KPZ equation. We show the direct relations between the cellular automata and KPZ equation. One model of these cellular automata is the time discrete version of the asymmetric simple exclusion process (ASEP) which belongs to KPZ universality class. The other model corresponds to the restricted solid on solid (RSOS) model which is introduced by Kim and Kosterlitz. [3]

This paper is organized as follows. In the section 2 we classify the equation into five regions in the parameter space and discuss the behavior of the solutions from the viewpoint of particles and anti-particles. In the section 3 we study the symmetry of the equation. We explain why the automata model represent the property of continuous Burgers equation. In the section 4 we discuss the stability of the equation. We study the meaning of the ultra-discrete limit. In the section 5 we construct the deterministic surface growth models. In the section 6 we introduce the noise into the discrete Burgers equation. In the section 7 we construct the noisy surface growth models using the ultra-discrete KPZ equation. The models are equivalent to the well known models which belong to the KPZ universality. The last section is devoted to the concluding remarks.

First we review the derivation of the ultra-discrete Burgers equation. Discretizing of both time and

space variables of diffusion equation, we can obtain

$$\frac{f_j^{t+1}-f_j^t}{\Delta T}=\frac{f_{j+1}^t-2f_j^t+f_{j-1}^t}{(\Delta X)^2}, \tag{1.3}$$

where ΔT and ΔX are lattice intervals in x and t respectively. Using the discrete analogue of Cole Hopf transformations

$$u_j^t=c\frac{f_{j+1}^t}{f_j^t}, \tag{1.4}$$

where c is constant. We can obtain the lattice version of the Burgers equation

$$u_j^{t+1}=u_{j-1}^t\frac{1+\frac{1-2\delta}{c\delta}u_j^t+\frac{1}{c^2}u_j^tu_{j+1}^t}{1+\frac{1-2\delta}{c\delta}u_{j-1}^t+\frac{1}{c^2}u_{j-1}^tu_j^t}, \tag{1.5}$$

where $\delta=\Delta T/(\Delta X)^2$.

We introduce a parameter ϵ and new variables

$$u_j^t=e^{U_j^t/\epsilon},\quad \frac{1-2\delta}{c\delta}=e^{-M/\epsilon},\quad c^2=e^{L/\epsilon}. \tag{1.6}$$

Then (1.5) becomes

$$\begin{aligned} U_j^{t+1}=U_{j-1}^t \quad &+\quad \epsilon\log[1+\exp(\frac{U_j^t-M}{\epsilon})+\exp(\frac{U_j^t+U_{j+1}^t-L}{\epsilon})] \\ &-\quad \epsilon\log[1+\exp(\frac{U_{j-1}^t-M}{\epsilon})+\exp(\frac{U_{j-1}^t+U_j^t-L}{\epsilon})] \end{aligned} \tag{1.7}$$

Here we take the so called "ultra-discrete limit"[6]

$$\epsilon\longrightarrow 0+. \tag{1.8}$$

Then we can obtain

$$U_j^{t+1}=U_j^t+\min(M,U_{j-1}^t,L-U_j^t)-\min(M,U_j^t,L-U_{j+1}). \tag{1.9}$$

Here we use the relation

$$\lim_{\epsilon\longrightarrow 0+}\epsilon\log(e^{\frac{A}{\epsilon}}+e^{\frac{B}{\epsilon}})=\max(A,B). \tag{1.10}$$

We call (1.9) the ultra-discrete Burgers equation.

2 Classification

(1.9) has a solution

$$U_j^t=\frac{L}{2}+\max(0,K(j+1)+\Omega t+\Theta_0)-\max(0,K(j)+\Omega t+\Theta_0), \tag{2.1}$$

with

$$\Omega=\begin{cases} |K| & \text{for } M\geq\frac{L}{2}, \\ \max(\frac{L}{2}-M,K,-K)-\frac{L}{2}+M & \text{for } M<\frac{L}{2}, \end{cases} \tag{2.2}$$

where Θ_0 is constant. This solution corresponds to the famous shock wave solution of the Burgers equation.Fig.1

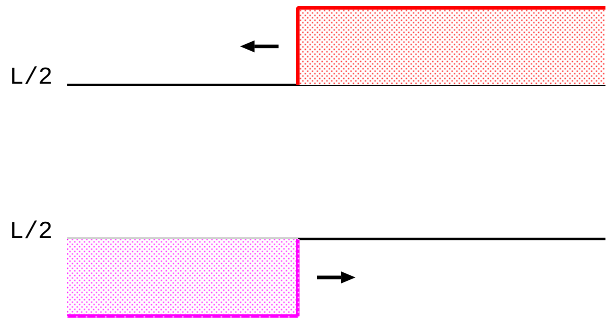

Figure 1: Shock wave solution

Here we classify five cases.

$$
\begin{aligned}
&(A1) \quad \frac{L}{2}+|K| \leq M, \quad (A2) \ \frac{L}{2} < M < \frac{L}{2}+|K|, \\
&(B) \quad \frac{L}{2} = M, \\
&(C1) \quad \frac{L}{2} > M > \frac{L}{2}-|K|, \quad (C2) \ \frac{L}{2}-|K| \geq M.
\end{aligned} \tag{2.3}
$$

At first we study the case (A) and (B). Note that in this case the dispersion relation is $\Omega = |K|$. The sign of K chooses the propagating direction and the velocity of the wave is -1 $(+1)$ for $K > 0$ $(K < 0)$. If we set $U_j^t \geq \frac{L}{2}$, then (1.9) becomes $U_j^{t+1} = U_{j+1}^t$. On th other hand, setting $U_j^t \leq \frac{L}{2}$, (1.9) becomes $U_j^{t+1} = U_{j-1}^t$. In both cases equations become linear. Then stable rectangular wave becomes a solution. We can write the rectangular solution explicitly as

$$
\begin{aligned}
U_j^t \quad = \quad & \frac{L}{2} + \max(0, K(j+1)+\Omega t+\Theta_0) - \max(0, K(j)+\Omega t+\Theta_0) \\
& - \max(0, K(j+1+l)+\Omega t+\Theta_0) + \max(0, K(j+l)+\Omega t+\Theta_0).
\end{aligned} \tag{2.4}
$$

The size of the rectangular is $K \times l$ Fig.2.

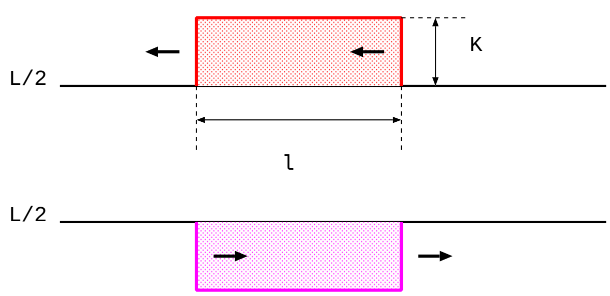

Figure 2: Rectangular solution

Here we introduce an interpretation of particles and anti-particles. We call that particles (anti-particles) are created at time t and j when $U_j^t > \frac{L}{2}$ $(U_j^t < \frac{L}{2})$. $U_j^t - \frac{L}{2}$ can be understood as the number of particles or an anti-particles. In this mean this particles or anti-particles are Boson. Then we call $U_j^t = \frac{L}{2}$ as vacuum state. From this interpretation this equation has chairarity. Particles go only to

the left and anti-particles go only to the right. If there are only particles (anti-particles), creation and annhiration of particles (anti-particles) do not occur. It can be seen from the result (2.4) is stable solution of (1.9).

We need to study interaction of particles and anti-particles. To consider interaction we consider collision of particles and anti-particles. Annihilation of particles and anti-particles occurs. If same numbers of particles and anti-particles meet, they become vacuum. The configuration of particles and anti-particles are in Fig.3. If the number of particles (anti-particles) are larger than the number of particles of anti-partials (particles), then only particles (anti-particles) whose number is difference of particles and anti-particles survive.

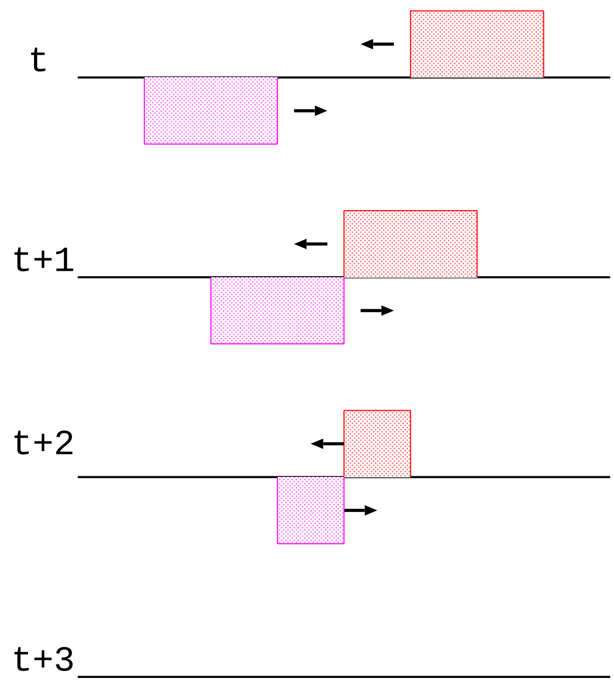

Figure 3: Collision of particles and anti-particles

Note that the point where particles and anti-particles meet does not move. It can be seen also in continuous limit as the collision of the shock waves. In the Burgers equation (1.1) setting $u(x,t) = u(x-vt)$, where v is propagation velocity. Imposing the boundary condition $u \to u_\pm$ and $\partial u \to 0$ for $x \to \pm\infty$, then we can obtain the soliton condition

$$u_+ + u_- = -v \tag{2.5}$$

In the limit $u_+ = u_-$ this condition implies $v = 0$. It means that the collision point does not move in the continuous Burgers equation. In fact the number of particles (anti-particles) is larger than that of anti-particles (particles), the collision point moves to the left (right) and the particles (anti-particles) only survive. It can be seen in the soliton condition (2.5).

The other configuration is in Fig.4. In the case (A) creation of pairs of particles and anti-particles

between particles and anti-particles occurs. On the other hand in the case (B) particles and anti-particles part over without creation of pairs of particles and anti-particles. It is the crucial difference between (A) and (B).

We consider the case same number K of particles and anti-particles part over. In the case (A1) K particles and K anti-particles are created. On the other hand in the case (A2) creation of $M-\frac{L}{2}$ pairs of particles and anti-particles occurs. If the number of particles are K_1 and anti-particles are K_2, in the case (A1) K pairs are created, in the case (A2)creation of $M-\frac{L}{2}$ pairs occurs, where $\min(K_1,K_2)=K$.

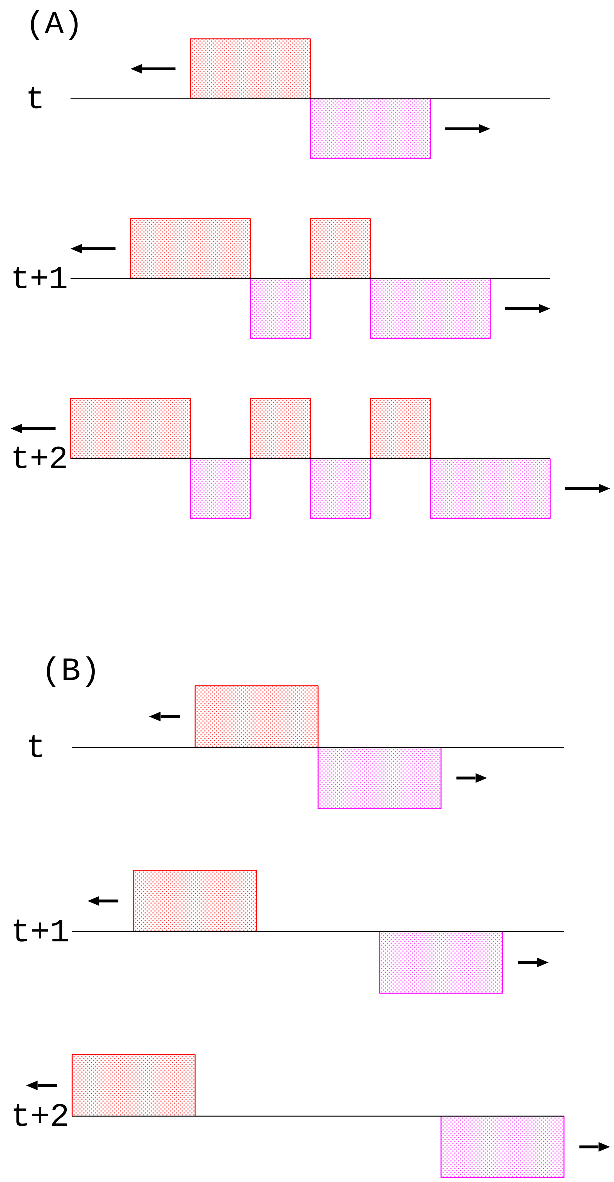

Figure 4: Separations of particles and anti-particles in the cases (A) and (B)

Next we consider the case (C). We can rewrite the dispersion relations

$$(C1) \quad \Omega = K + M - \frac{L}{2}, \quad (C2) \quad \Omega = 0. \tag{2.6}$$

The characteristic is the dispersion relations with a gap in the case (C1). From (2.6) in the case (C2) the shock wave solution (2.1) does not move. In other words $U_j^{(t+1)} = U_j^{(t)}$. Then any initial configurations of particles and anti-particles does not move. In the case (C1) the speed of the shock wave solution is

$$v = \frac{|K| + M - \frac{L}{2}}{K}, \tag{2.7}$$

where v is the speed of the shock wave. The shock wave with positive parity, $K > 0$, propagates left with negative velocity, whereas the shock wave with opposite parity, $K < 0$, propagate in the forward

direction as we have shown in the cases (A) and (B). However in the case (C1) the speed v has a range. The maximum speed is 1 or -1. Notice that in the limit $K \to \pm\infty$ the speed of the shock wave become ± 1. If the shock waves with the positive parity and negative parity collide, the annihilation of the particles and anti-particles occurs as we have seen in the cases (A) and (B). The difference is the speed of the annihilation. In the case (C1) there are not rectangular solutions. Then the pair creation of particles and anti-particles does not occur. From these results the case (C1) represents the property of the continuous Burgers equation.

3 Symmetry of Equations

Here we note that the Burgers equation (1.1) is invariant under the parity transformation $x \to -x$ provided $u \to -u$. This feature is related to the presence on a single spatial derivative in the derivative term. We can not see this parity invariant in the discrete version of the Burgers equation. We consider the other ultra-discrete limit

$$\epsilon \longrightarrow -0. \tag{3.1}$$

In this limit we use a following relation

$$\lim_{\epsilon \to 0-} \epsilon \log(e^{\frac{A}{\epsilon}} + e^{\frac{B}{\epsilon}}) = \min(A, B). \tag{3.2}$$

Using this relation we can take the other ultra-discrete limit. Then we can obtain

$$\begin{aligned} U_j^{t+1} &= U_j^t + \max(M, U_{j-1}^t, L - U_j^t) - \max(M, U_j^t, L - U_{j+1}^t) \\ &= U_j^t + \min(\tilde{M}, U_{j+1}^t, \tilde{L} - U_j^t) - \min(\tilde{M}, U_j^t, L - U_{j-1}^t), \end{aligned} \tag{3.3}$$

where $\tilde{L} = L$ and $\tilde{M} = L - M$. (3.3) can be obtained using the transformations of (1.9)

$$j + 1 \longrightarrow j - 1, \quad j - 1 \longrightarrow j + 1, \quad M \longrightarrow \tilde{M}, \quad L \longrightarrow \tilde{L}. \tag{3.4}$$

This is the parity transformation. (3.4) corresponds to $x \to -x$ in the continuous case. (3.1) is the transformation $U_j^t \longrightarrow -U_j^t$. Then the invariance under parity transformation is recovered. The particles move only to the right and the anti-particles only to the left for (3.3). The particles and anti-particles move to the opposite directions to (1.9).

Here we consider why the parity invariance can be seen in the continuous case (1.1) and ultra-discrete limit (1.9), but not be seen in the full discrete type euqtaion (1.5). The relation between the discrete variables and the continuous variables are following

$$u(j\Delta x, t\Delta T) = \frac{1}{\Delta X} \log \frac{u_j^t}{c}. \tag{3.5}$$

Using (1.8), we can obtain a relation

$$u(j\Delta x, t\Delta T) = \frac{1}{\Delta X \epsilon}(U_j^t - \frac{L}{2}). \tag{3.6}$$

The vacuum state $U_j^t = L/2$ corresponds to $u = 0$. It can be seen in the collision of the shocks.

Using the relation (3.5) and (3.6) the relation among continuous, discrete and ultra discrete variables are

$$v \sim \log u_j^t \sim U_j^t. \tag{3.7}$$

Then the the parity invariance is the same in the continuous (1.1) and ultra-discrete Burgers equations (1.9).

4 Stability

Here we consider stability of the solution of the discrete Burgers equation (1.5). We substitute plane wave solution $u_j^t = e^{\mathrm{i}jk+\omega t}$ into the discrete diffusion equation (1.3). We can obtain the condition for the stability condition $|e^{\omega}| \le 1$,

$$\delta = 0. \tag{4.1}$$

Using the second and the third equations of (1.8) we can get the relation

$$\delta = \frac{1}{2 + \exp[\frac{1}{\epsilon}(\frac{L}{2} - M)]} \tag{4.2}$$

Using this relation, the cases (A), (B) and (C) correspond to $\delta = 1/2$, $\delta = 1/3$ and $\delta = 0$ respectively.Fig.5. From the condition (4.1) the cases (A) and (B) are unstable and (C) is only stable in the discrete Burgers equation. It is the reason why the case (C) represents the property of the continuous Burgers equation (See section 2). It also can be seen that we need to set $\delta \to 0$ to obtain the continuous Burgers equation (1.1) from the lattice Burgers equation (1.5). On the other hand there is the rectangular solutions in the cases (A) and (B) which can not be seen in the Burgers equation. The rectangular solutions of (A) and (B) correspond to solutions of configurations of two solitons with opposite parity in the continuous Burgers equation. To construct this configuration we need the condition that the solitons are well separated and non overlapping. Then the size of the solutions are $\infty \times \infty$. They are rescaled and becomes the rectangular solutions of the cases (A) and (B) in the ultra discrete limit.

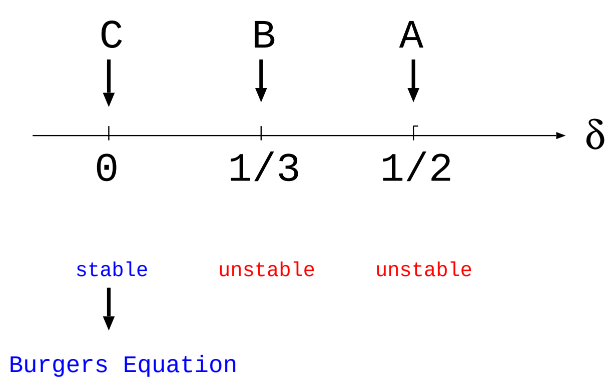

Figure 5: Stability of discrete Burgers equation

5 Deterministic ultra-discrete KPZ equation

If we set $u(x,t) = (\partial/\partial x)h(x,t)$, $h(x,t)$ is governed by the deterministic KPZ equation. To construct automata model for deterministic KPZ equation, from (3.7) we can set

$$U_j^t = H_{j+1}^t - H_j^t + \frac{L}{2}. \tag{5.1}$$

The last term is selected the constant for the vacuum state. We rewrite (1.9)

$$H_j^{t+1} = \max(H_{j-1}^t, H_j^t + \frac{L}{2} - M, H_{j+1}^t) + C, \tag{5.2}$$

where C is a constant. In the same way we can rewrite (3.3) by the independent variables H_j^t,

$$H_j^{t+1} = \min(H_{j-1}^t, H_j^t + \frac{\tilde{L}}{2} - \tilde{M}, H_{j+1}^t) + \tilde{C}, \tag{5.3}$$

where $\tilde{C}$ is a constant. We call these equations (5.2) and (5.3) "ultra-discrete deterministic KPZ equation."

To describe interface dynamics we consider the periodic condition

$$H_1^t = H_N^t. \tag{5.4}$$

Here H_i^t denotes the height of the surface at the lattice site i at the integer time t. Solutions of ultra-discrete KPZ equation are classed by (A1)-(C2) as the ultra-discrete Burgers equation.

We consider the deterministic dynamical model. In this interface model the integer height variable H_j^t may differ on neighbor sites j and $j+1$ only ± 1. The two elementary steps, deposition and evaporation, which define the surface evolution are illustrated in Fig.6. In a lattice gas language the height differences between neighorbouring sites are mapped to a particle occupation number $n_x^t = 0, 1$ with the presence of a particle on x corresponding to slope -1 between sites $j-1$ and j in the interface model and a vacancy at site x corresponding to slope $+1$. The particles have a simple dynamics. With rate 1 they jump to the right (left) except when the final site is occupied, in which case they stay (hard core exclusion). In this model growth (evaporate) can occur in local minima (maxima). These models are deterministic. The configuration at time $t+1$ is determined from the configuration at time t by a local rule which depends on nearest neighbors only. The corresponding lattice gas evolves according to the automaton rule 184.[8]

In the case (A1) we set $K = \pm 1$, $C = -1$, $\tilde{C} = 1$, in (5.2) and (5.3) describe following deterministic model,

$$H_j^{t+1} = \max(H_{j-1}^t, H_{j+1}^t) - 1, \tag{5.5a}$$

and

$$H_j^{t+1} = \min(H_{j-1}^t, H_{j+1}^t) + 1. \tag{5.5b}$$

The model particles move only to the right corresponds to (5.5a) and the model only to the left is (5.5b). The above deterministic dynamical model is equivalent to the ultra-discrete KPZ equation (5.5a)

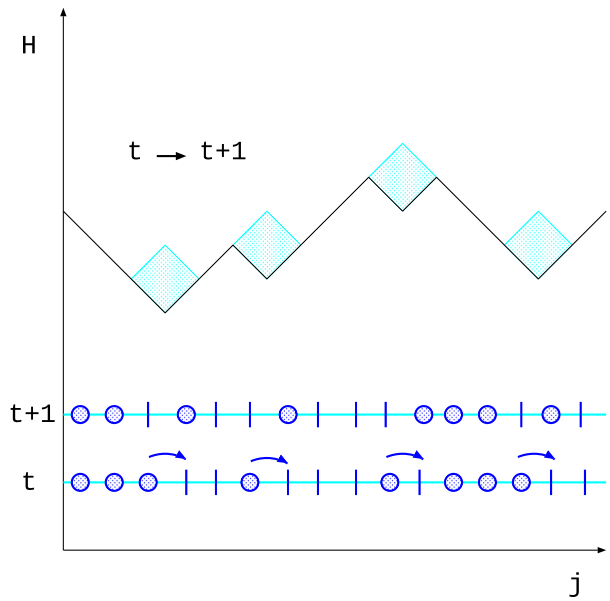

Figure 6: Deterministic Surface Growth model A

or (5.5b). (5.5a) is the surface evaporate model and (5.5b) is the surface growth model. We call these models “A model”.

We consider another interface model. In this interface model the integer height variable H_j^t may differ on neighbor sites j and $j+1$ only ± 1or 0. Consider a one dimensional surface configuration parallel to the j axis of a square lattice.

In the case (B) we set $K = \pm 1, 0$, $C = 0$, in (5.2) describe following deterministic model,

$$H_j^{t+1} = \max(H_{j-1}^t, H_j^t, H_{j+1}^t). \tag{5.6}$$

The deterministic model (3) was studied in [7].

In the case (B) we set $K = \pm 1, 0$, $C = -1$ and $\tilde{C} = 1$, in (5.2) and (5.3)

$$H_j^{t+1} = \max(H_{j-1}^t, H_j^t, H_{j+1}^t) - 1. \tag{5.7a}$$

and

$$H_j^{t+1} = \min(H_{j-1}^t, H_j^t, H_{j+1}^t) + 1. \tag{5.7b}$$

In the model (5.7a) a particle is evaporated at the local maxima and in the model (5.7b) a particle is deposited at the local minima. The dynamical equations (5.7a) and (5.7b) restrict the height difference ± 1 or 0 automatically. We call the models (5.7a) and (5.7b) “ B model ”.

In the case (C2) we also consider as the RSOS model. If we we set $K = \pm 1, 0$, $C = -1$ and $\tilde{C} = 1$, in (5.2) and (5.3), we can obtain the model whose dynamics is trivial $H_j^{t+1} = H_j^t + 1$ or $H_j^{t+1} = H_j^t - 1$.

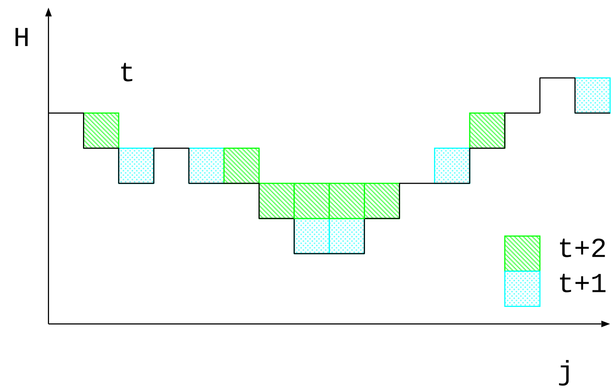

Figure 7: Deterministic Surface Growth model B

6 Ultra-discretization with noise

Next we consider the discrete Burgers equation with noise,

$$u_j^{t+1} = u_j^t \frac{\frac{1}{u_j^t} + \frac{1-2\delta}{c\delta} + \frac{1}{c^2}u_{j+1}^t + \frac{1}{c^2}\xi_{j+1}^t}{\frac{1}{u_{j-1}^t} + \frac{1-2\delta}{c\delta} + \frac{1}{c^2}u_j^t + \frac{1}{c^2}\xi_j^t}, \tag{6.1}$$

where ξ_j^t is the noise. If we set $\xi_j^t = 0$, (6.1) is the discrete Burgers equation (1.5). In the limit $\Delta X \to 0$ and $\Delta T \to 0$ with $\delta \to 0$, (6.1) becomes the noisy Burgers equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x} + \frac{\partial \xi(x,t)}{\partial x}, \tag{6.2}$$

where the white noise $\xi(x,t)$

$$< \xi(x,t)\xi(x',t') >= \mathrm{D}\delta(x,x')\delta(t,t'). \tag{6.3}$$

The relation between the discrete noise and the continuous noise is

$$\xi(x,t) = \frac{1}{(\Delta X)^2}\log \xi_j^t. \tag{6.4}$$

We introduce a parameter ϵ and new variables

$$\xi_j^t = e^{X_j^t/\epsilon}. \tag{6.5}$$

We take the ultra-discrete limit (1.8) in the discrete noisy Burgers equation (6.1). We can obtain

$$U_j^{t+1} = U_j^t + \max(-M, -U_j^t, U_j^t - L, X_{j+1}^t - \frac{L}{2}) - \max(-M, -U_{j-1}^t, U_j - L, X_j - \frac{L}{2}). \tag{6.6}$$

7 Ultra-discrete KPZ equation

We rewrite this equation using the independent variables H_j^t (5.1)

$$H_j^{t+1} = \max(H_{j-1}^t, H_j^t + \frac{L}{2} - M, H_{j+1}^t, H_j^t + X_j^t) + C, \tag{7.1}$$

where C is is a constant. This is the noisy version of the model (5.2). In the same way we can obtain

$$H_j^{t+1} = \min(H_{j-1}^t, H_j^t + \frac{\tilde{L}}{2} - \tilde{M}, H_{j+1}^t, H_j^t + \tilde{X}_j^t) + \tilde{C}, \tag{7.2}$$

where $\tilde{C}$ is is a constant. This is the noisy version of the model (5.3).

Here we set $K = \pm 1$, $\tilde{C}_1 = -1$, and $\tilde{C}_2 = 1$, we can obtain

$$H_j^{t+1} = \max(H_{j-1}^t, H_{j+1}^t, H_j^t + X_j^t) - 1, \tag{7.3a}$$

and

$$H_j^{t+1} = \min(H_{j-1}^t, H_{j+1}^t, H_j^t + \tilde{X}_j^t) + 1. \tag{7.3b}$$

These are the noisy version of the A models (5.5a) and (5.5b). Here we set the noise as following

$$X_j^t = \tilde{X}_j^t = \begin{cases} 1 & \text{probability } q \\ -1 & \text{probability } 1-q \end{cases} \tag{7.4}$$

In (7.3a) and (7.3b) we consider the non-deterministic dynamical model. In the lattice gas language the particles have a simple dynamics. With rate q they jump to the right except when the final site is occupied for (7.3a). For (7.3b) with rate q they jump to the left. In (7.3a) for t time steps the probability distribution that a particle moves n space steps is the binomial distribution

$$P(t,n) = \begin{pmatrix} t \\ n \end{pmatrix} q^n (1-q)^{t-n}. \tag{7.5}$$

It is well known that in the large t limit with $\lambda = qt$ =const, we can obtain the Poisson distribution. Then in the continuous limit of time we can obtain the Poisson process.

A lattice model of particles moving stochastically with hard-core exclusion is called the asymmetric simple exclusion process (ASEP).[9] Each particle hops to the right (left) nearest site with the probability $p_1 \mathrm{d}t$ ($p_2 \mathrm{d}t$) in every infinitesimal time interval $\mathrm{d}t$. It is a Poisson process. (7.3a) corresponds to the time discrete version of ASEP with $p_1 = 0$ and (7.3b) corresponds to the ASEP with $p_2 = 0$.

For the B model we set $K = \pm 1, 0$, $\tilde{C}_1 = -1$, and $\tilde{C}_2 = 1$, we can obtain

$$H_j^{t+1} = \max(H_{j-1}^t, H_j^t, H_{j+1}^t, H_j^t + X_j^t) - 1, \tag{7.6a}$$

and

$$H_j^{t+1} = \min(H_{j-1}^t, H_j^t, H_{j+1}^t, H_j^t + \tilde{X}_j^t) + 1, \tag{7.6b}$$

In the model (7.6a) a particle is evaporated at the local maxima with the rate q and in the model (7.6b) a particle deposited at the local minima with the rate $1-q$.

In the RSOS model introduced by Kim and Kosterlitz (KK),[3] a particle is deposited at randomly selected site as long as the height difference Δh between nearest-neighbor columns remains as $h \leq 0$. Then a particle is deposited in the local minima. The difference between our B model and the KK model is time steps. In the B model in one time step m particles are deposited. It corresponds to m time steps in KK model. The difference does not affect the universality.

8 Concluding Remarks

We consider the ultra-discrete Burgers equation. We classify the equation into five regions in the parameter space. We discussed the behavior of the solutions. The A and B models have the rectangular solutions. On the other hand the C model has only shock wave solutions. We construct the deterministic surface growth models using A and B models. Furthermore we introduce the noise into the ultra discrete Burgers equation. We present the automata models of KPZ equation. The A model becomes the discrete time version of ASEP and B model corresponds to the KK model. It is well known ASEP and KK model belong to the universality of KPZ equation. We discussed the symmetry of the ultra-discrete and continuous Burgers equation. We showed why automata model represent the property of continuous Burgers equation. Furthermore we study the stability of equations. Using this result we can understand the parts of the meaning of ultra-discrete limit.

We can extend these method to the higher dimension and we can introduce the quenched noise into the ultra-discrete KPZ equation. About these contents we will discuss else where.

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